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Positive Definite Kernels on Complex Spheres

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Continuous bizonal positive definite kernels on the spheres in $\mathbb{C}^q$ are shown to be a series of disk polynomials with nonnegative coefficients. These kernels are the complex analogs of the so-called positive definite functions on real spheres introduced and characterized by I. J. Schoenberg (1942, *Duke Math. J.* **9**, 96–108). The result adds to the classes of functions from which one can generate radial basis function interpolants to arbitrary data on spheres. © 2001 Academic Press

1. INTRODUCTION

We are interested in positive definite kernels on the unit sphere Ω_{2q} in $\mathbb{C}^q$ with emphasis on those continuous ones having the form

$$(\xi, \zeta) \in \Omega_{2q}^2 \to f(\langle \xi, \zeta \rangle).$$

In this expression, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{C}^q$ and f is a complex function defined on Ω_2 when $q = 1$ and on the closed unit disk B_2 in $\mathbb{C}$ otherwise. Kernels of the above form are called *bizonal* due to the fact that they become zonal functions on Ω_{2q} when one of the variables is fixed. A function is *ζ-zonal* if it is of the form $\xi \in \Omega_{2q} \to g(\langle \xi, \zeta \rangle)$

for some function g defined on B_2 (Ω_2, if $q = 1$). This concept agrees with that appearing in many books dealing with analysis on real spheres [7, 10, 13].

We recall that a kernel $K: A^2 \to \mathbb{C}$ is *positive definite* on a nonempty set A if and only if

$$\sum_{\mu, \nu = 1}^{l} c_\mu \overline{c_\nu} K(\xi_\mu, \xi_\nu) \geq 0,$$

for all $l \in \mathbb{N} := \{1, 2, \ldots\}$, $\{\xi_1, \xi_2, \ldots, \xi_l\} \subset A$ and $\{c_1, c_2, \ldots, c_l\} \subset \mathbb{C}$. In other words, K is positive definite on A if and only if every matrix of the form $(K(\xi_\mu, \xi_\nu)), \{\xi_1, \xi_2, \ldots, \xi_l\} \subset A$, is nonnegative definite. It is convenient to call a function f *positive definite* on $A \subset \Omega_{2q}$ whenever the corresponding bizonal kernel $f(\langle \cdot, \cdot \rangle)$ is positive definite on A.

Our primary interest in bizonal kernels is due to the possibility of using them to perform interpolation on Ω_{2q}. Under reasonable additional hypotheses on the positive definite function f on Ω_{2q}, it is possible to guarantee the positive definiteness of many matrices of the form $(f(\langle \xi_\mu, \xi_\nu \rangle))$ and as a consequence interpolation of arbitrary data at the ξ_μ by a function of the form

$$\xi \in \Omega_{2q} \to \sum_{\mu = 1}^{l} c_\mu f(\langle \xi, \xi_\mu \rangle),$$

can be performed. When the points ξ_μ belong to a real sphere contained in Ω_{2q} then the restriction of the above interpolant to that sphere becomes a spherical radial basis function or spherical spline; the reader is referred to [3, 6] and references therein for acquaintance with this recent and important field of research in approximation theory.

The main purpose of this paper is to present a complete characterization of continuous bizonal positive definite kernels on Ω_{2q} thus generalizing a famous result of I. J. Schoenberg [11] on positive definite functions on real spheres. In addition to that, the result expands the class of functions that can be useed to construct spherical radial basis function interpolants to arbitrary data on real spheres. Our approach has similarities with that of Schoenberg but the arguments here are more complicated. In the present situation, we will deal with a class of homogeneous, zonal, and harmonic polynomials in $\mathbb{C}^q$ and with the Poisson–Szegő kernel for the unit ball B_{2q} of $\mathbb{C}^q$. The above mentioned polynomials are connected to the so-called disk polynomials which are two dimensional analogs of Jacobi polynomials [1, 9] while the Poisson–Szegő kernel generates solutions to certain Dirichlet problems on B_{2q} and is connected to Jacobi polynomials and certain hypergeometric functions [12].

The paper is organized as follows. In Section 2, we introduce notation and present necessary technical lemmata. In Section 3, we characterize positive definite bizonal kernels on Ω_2. In Section 4, taking Section 3 as a model, we characterize the positive definite bizonal kernels on Ω_{2q}, $q \geq 2$. At last, in Section 5, we make a brief discussion on the connection between positive definite functions on Ω_{2q} and on real spheres.

2. PRELIMINARY AND BASIC LEMMATA

The Laplace–Beltrami operator Δ_{2q} associated to the Bergman metric on B_{2q} reveals itself in the field of several complex variables and is the basic invariant differential operator on the symmetric space isomorphic to B_{2q}. It can be defined by

$$\Delta_{2q} = \frac{4}{q+1}(1 - |z|^2) \sum_{\mu,\nu=1}^{q} \left(\delta_{\mu\nu} - z_\mu \overline{z_\nu}\right) \frac{\partial^2}{\partial z_\mu \, \partial \overline{z_\nu}},$$

$$z = (z_1, z_2, \ldots, z_q) \in B_{2q},$$

and it is usually attached to the Poisson–Szegő kernel on Ω_{2q}, i.e., the kernel

$$\mathscr{P}_q(z, \zeta) := \frac{(q-1)!}{2\pi^q} \frac{\left(1 - |z|^2\right)^q}{|1 - \langle z, \zeta \rangle|^{2q}}, \qquad (z, \zeta) \in B_{2q} \times \Omega_{2q}.$$

Together, they appear as the main subject in the following Dirichlet problem [12]:

LEMMA 2.1. *Given a continuous function f: $\Omega_{2q} \to \mathbb{C}$, there exists a continuous function u: $B_{2q} \to \mathbb{C}$ such that $\Delta_{2q} u = f$ and $u|\Omega_{2q} = f$. The solution u can be computed by the formula*

$$u(z) = \int_{\Omega_{2q}} \mathscr{P}_q(z, \zeta) f(\zeta)\, d\omega_{2q}(\zeta), \qquad z \in B_{2q},$$

in which $d\omega_{2q}$ denotes the rotation-invariant surface element on Ω_{2q}.

Some additional properties of the Poisson–Szegő kernel are $\mathscr{P}_q \geq 0$, $\mathscr{P}_q(\cdot, \zeta)$ is annihilated by Δ_{2q}, and $\mathscr{P}_q$ coincides with the standard Poisson kernel when $q = 1$. If the inner product between square-integrable functions on Ω_{2q} is defined by

$$\langle\langle f, g \rangle\rangle = \int_{\Omega_{2q}} f(\zeta) \overline{g(\zeta)}\, d\omega_{2q}(\zeta),$$

then the following orthogonal decomposition holds

$$L^2(\Omega_{2q}, d\omega_{2q}) = \bigoplus_{m,n=0}^{\infty} \mathscr{H}^q_{m,n},$$

in which $\mathscr{H}^q_{m,n}$ is the space of restrictions to Ω_{2q} of harmonic polynomials in the variables z and $\bar{z}$ on $\mathbb{C}^q$, that are homogeneous of degree m in z and degree n in $\bar{z}$. Here, a harmonic function is understood to be one that is annihilated by the complex Laplacian

$$\sum_{j=1}^{q} \frac{\partial^2}{\partial z_j \, \partial \overline{z_j}}, \qquad z = (z_1, z_2, \ldots, z_q).$$

Several classes of disk polynomials play an important role in the spaces just described. Recall that the *disk polynomial* of degree $m+n$ in x and y associated to a positive real number α is the polynomial $R^{\alpha}_{m,n}$ given by

$$R^{\alpha}_{m,n}(z) := r^{|m-n|} e^{i(m-n)\theta} R^{(\alpha,|m-n|)}_{m\wedge n}(2r^2 - 1), \qquad z = re^{i\theta} = x + iy,$$

in which $R^{(\alpha,|m-n|)}_{m\wedge n}$ is the usual Jacobi polynomial of degree $m \wedge n := \min\{m, n\}$ associated to the numbers α and $|m-n|$, and normalized by $R^{(\alpha,|m-n|)}_{m\wedge n}(1) = 1$. It is easy to see that $R^{\alpha}_{m,n}$ is a polynomial of degree m in the variable z and of degree n in the variable $\bar{z}$. Due to the orthogonality relations for Jacobi polynomials, the set $\{R^{\alpha}_{m,n}\colon 0 \le m,\ n < \infty\}$ is a complete orthogonal system in $L^2(B_2, dw_\alpha)$, where dw_α is the positive measure of total mass one on B_2 given by

$$dw_\alpha(z) = \frac{\alpha+1}{\pi}\left(1 - x^2 - y^2\right)^{\alpha} dx\, dy, \qquad z = x + iy.$$

For further information on disk polynomials see [2, 4, 9] and several references given there.

The isomorphism between Ω_{2q} and the homogeneous space $U(q)/U(q-1)$, the quotient of the group $U(q)$ of unitary transformations of $\mathbb{C}^q$ by its subgroup $U(q-1)$ composed of unitary transformations that leave a "pole" of Ω_{2q} fixed, is the doorway for a beautiful theory in which disk polynomials play the role played by ultraspherical polynomials in harmonic analysis on real spheres [1, 8, 9]. The connection between disk polynomials and the spaces $\mathscr{H}^q_{m,n}$ is made by the following result implicit in [9].

LEMMA 2.2. *The ζ-zonal functions in $\mathscr{H}^q_{m,n}$, $q \ge 2$, are the constant multiples of the function $\xi \in \Omega_{2q} \to R^{q-2}_{m,n}(\langle \xi, \zeta \rangle)$.*

If $\{Y^q_1, Y^q_2, \ldots, Y^q_{N(q;m,n)}\}$ is an orthonormal basis for $\mathscr{H}^q_{m,n}$ with respect to the inner product previously defined then the following nice addition formula holds [9]:

$$R^{q-2}_{m,n}(\langle \xi, \zeta \rangle) = \frac{\omega_{2q}}{N(q;m,n)} \sum_{j=1}^{N(q;m,n)} Y_j^q(\xi)\overline{Y_j^q(\zeta)}, \qquad \xi, \zeta \in \Omega_{2q}.$$

In this expression, ω_{2q} is the total surface of Ω_{2q}, i.e., $\omega_{2q} = 2\pi^q/(q-1)!$. Among other things this formula reveals that $R^{q-2}_{m,n}$ is positive definite on Ω_{2q}. Indeed, by the use of the addition formula we have that

$$\begin{aligned}
&\sum_{\mu=1}^{l}\sum_{\nu=1}^{l} c_\mu \overline{c_\nu} R^{q-2}_{m,n}\big(\langle \xi_\mu, \xi_\nu \rangle\big) \\
&\quad = \sum_{\mu=1}^{l}\sum_{\nu=1}^{l} c_\mu \overline{c_\nu} \frac{\omega_{2q}}{N(q;m,n)} \sum_{j=1}^{N(q;m,n)} Y_j^q(\xi_\mu)\overline{Y_j^q(\xi_\nu)} \\
&\quad = \frac{\omega_{2q}}{N(q;m,n)} \sum_{j=1}^{N(q;m,n)} \sum_{\mu=1}^{l} c_\mu Y_j^q(\xi_\mu) \sum_{\nu=1}^{l} \overline{c_\nu}\, \overline{Y_j^q(\xi_\nu)} \\
&\quad = \frac{\omega_{2q}}{N(q;m,n)} \sum_{j=1}^{N(q;m,n)} \left| \sum_{\mu=1}^{l} c_\mu Y_j^q(\xi_\mu) \right|^2 \geq 0,
\end{aligned}$$

whenever $l \geq 1$, $\{\xi_1, \xi_2, \dots, \xi_l\} \subset \Omega_{2q}$ and $\{c_1, c_2, \dots, c_l\} \subset \mathbb{C}$.

The last property of disk polynomials needed in this paper is the main result in [5], a series representation for the Poisson–Szegő kernel in terms of disk polynomials.

Lemma 2.3. *If $q \geq 2$ then*

$$\mathscr{P}_q(r\xi, \zeta) = \sum_{(m,n)\in\mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} S^q_{m,n}(r) R^{q-2}_{m,n}(\langle \xi, \zeta \rangle),$$
$$0 \leq r < 1,\ \xi, \zeta \in \Omega_{2q},$$

in which $S^q_{m,n}(r) \geq 0$ and $\lim_{r\to 1^-} S^q_{m,n}(r) = 1$. The series converges absolutely and uniformly for $\xi, \zeta \in \Omega_{2q}$ and $0 \leq r \leq \rho$, for each $\rho < 1$.

We observe that the function $S^q_{m,n}(r)$ is in fact a multiple of the product of r^{m+n} by the hypergeometric function $F(m, n, m+n+q, r^2)$ associated to the triple $(m, n, m+n+q)$, as explained in [5].

We end this section with a consequence of the definition of positive definiteness on spheres to be used in the next two sections. The first step to reach it is to obtain a complex version of a well-known result found by W. H. Young a long time ago [14].

LEMMA 2.4. *A continuous function f is positive definite on Ω_{2q} only if*

$$\int_{\Omega_{2q}}\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) g(\xi) \overline{g(\zeta)}\, d\omega_{2q}(\xi)\, d\omega_{2q}(\zeta) \geq 0,$$

for every continuous function $g\colon \Omega_{2q} \to \mathbb{C}$.

Proof. Let f be positive definite on Ω_{2q} and g a continuous function on Ω_{2q}. Since the inequality

$$\sum_{\mu, \nu=1}^{l} c_\mu \overline{c_\nu} f\big(\langle \xi_\mu, \xi_\nu \rangle\big) \geq 0,$$

holds for every positive integer l, $\{\xi_1, \xi_2, \ldots, \xi_l\} \subset \Omega_{2q}$ and $\{c_1, c_2, \ldots, c_l\} \subset \mathbb{C}$, we take $l \geq 2$, $c_\mu = g(\xi_\mu)$, $\mu = 1, 2, \ldots, l$, and integrate with respect to ξ_μ and ξ_ν. From each of the diagonal terms we obtain

$$\omega_{2q} f(1) \int_{\Omega_{2q}} |g(\xi_\mu)|^2 d\omega_{2q}(\xi_\mu),$$

while each of the terms for which $\mu \neq \nu$ yields

$$\int_{\Omega_{2q}}\int_{\Omega_{2q}} f\big(\langle \xi_\mu, \xi_\nu \rangle\big) g(\xi_\mu) \overline{g(\xi_\nu)}\, d\omega_{2q}(\xi_\mu)\, d\omega_{2q}(\xi_\mu).$$

Thus,

$$l\omega_{2q} f(1) \int_{\Omega_{2q}} |g(\xi)|^2 d\omega_{2q}(\xi)$$
$$+\, l(l-1) \int_{\Omega_{2q}}\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) g(\xi) \overline{g(\zeta)}\, d\omega_{2q}(\xi)\, d\omega_{2q}(\zeta) \geq 0.$$

The continuity of g implies that the single integral above is finite. Hence, on dividing by $l(l-1)$ and letting l go to infinity, we deduce that

$$\int_{\Omega_{2q}}\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) g(\xi) \overline{g(\zeta)}\, d\omega_{2q}(\xi)\, d\omega_{2q}(\zeta) \geq 0,$$

which is the desired inequality. ∎

We shall use the following weak version of the previous lemma.

COROLLARY 2.5. *A continuous function f is positive definite on Ω_{2q} only if*

$$\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle)\, d\omega_{2q}(\xi) \geq 0, \qquad \zeta \in \Omega_{2q}.$$

Proof. First, take g equal to a constant in the previous theorem to conclude that a positive definite function f on Ω_{2q} satisfies

$$\int_{\Omega_{2q}}\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle)\, d\omega_{2q}(\xi)\, d\omega_{2q}(\zeta) \geq 0.$$

Next, observe that to conclude the proof it suffices to show that the following invariance property holds:

$$\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle)\, d\omega_{2q}(\xi) = \int_{\Omega_{2q}} f(\langle \xi, \eta \rangle)\, d\omega_{2q}(\xi), \qquad \zeta, \eta \in \Omega_{2q}.$$

Since this is certainly true in the case $q = 1$, we assume that $q \geq 2$. By decomposing the point ξ in the form

$$\xi = \cos\phi e^{i\varphi}\zeta + \sin\phi\, \xi', \qquad \xi' \in \Omega_{2q-2},\ \phi \in (0, 2\pi),\ \varphi \in \mathbb{R} \bmod 2\pi,$$

we see that

$$d\omega_{2q}(\xi) = \cos\phi(\sin\phi)^{2q-3}\, d\phi\, d\varphi\, d\omega_{2q-2}(\xi').$$

Making the changes $r = \cos\phi$ and $\varphi = -\theta$, we reach

$$d\omega_{2q}(\xi) = r(1 - r^2)\, dr\, d\theta\, d\omega_{2q-2}(\xi'),$$

and, consequently,

$$\int_{\Omega_{2q}} f(\langle \xi, e^{i\varphi}\zeta \rangle)\, d\omega_{2q}(\xi) = 2\pi\omega_{2q-2}\int_{-1}^{1} r(1 - r^2)^{q-2} f(r)\, dr,$$

which is independent of φ and ζ. Finally, taking an element B of $U(q)$ such that $B(e^{i\varphi}\zeta) = \zeta$ and using the invariance of $d\omega_{2q}$ with respect to elements of $U(q)$, we obtain

$$\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle)\, d\omega_{2q}(\xi) = \int_{\Omega_{2q}} f(\langle \xi, e^{i\varphi}\zeta \rangle)\, d\omega_{2q}(\xi).$$

The result follows. ∎

3. ZONAL POSITIVE DEFINITE KERNELS ON Ω_2

The starting point in this section is the Poisson–Szegő kernel on Ω_2. Since it coincides with the Poisson kernel, we have [12].

$$\mathscr{P}_1(z, \zeta) = \frac{1 - |z|^2}{2\pi|1 - \langle z, \zeta \rangle|^2}, \qquad (z, \zeta) \in B_2 \times \Omega_2.$$

Writing z in polar form the previous equation reads

$$\mathscr{P}_1(r\xi, \zeta) = \sum_{k \in \mathbb{Z}} r^{|k|} \langle \xi, \zeta \rangle^k, \qquad \xi, \zeta \in \Omega_2.$$

The solution of the Dirichlet problem on Ω_2 given in Lemma 2.1 can now be expressed in the form

$$\int_{\Omega_2} \mathscr{P}_1(r\xi, \zeta) f(\zeta)\, d\omega_2(\zeta)$$

$$= \sum_{k \in \mathbb{Z}} r^{|k|} \int_{\Omega_2} f(\zeta) \langle \xi, \zeta \rangle^k d\omega_2(\zeta), \qquad \xi \in \Omega_2.$$

Recalling Lemma 2.1, we now see that a continuous function $f\colon \Omega_2 \to \mathbb{C}$ has a series representation in the form

$$f(\xi) \sim \sum_{k \in \mathbb{Z}} \int_{\Omega_2} f(\zeta) \langle \xi, \zeta \rangle^k d\omega_2(\zeta), \qquad \xi \in \Omega_2.$$

The use of this representation and of Corollary 2.5 yields our next result.

THEOREM 3.1. *A continuous function f is positive definite on Ω_2 if and only if it is representable in the form*

$$f(\xi) = \sum_{k \in \mathbb{Z}} a_k \xi^k, \qquad \xi \in \Omega_2,$$

in which $\sum_{k \in \mathbb{Z}} a_k < \infty$ and $a_k \geq 0$ for all k.

Proof. Let f be positive definite on Ω_2. For $\eta \in \Omega_2$, the function $f(\langle \cdot, \eta \rangle)$ can be represented in the form

$$f(\langle \xi, \eta \rangle) \sim \sum_{k \in \mathbb{Z}} \int_{\Omega_2} f(\langle \zeta, \eta \rangle) \langle \eta, \zeta \rangle^k \langle \xi, \eta \rangle^k d\omega_2(\zeta)$$

$$= \sum_{k \in \mathbb{Z}} \left(\int_{\Omega_2} f(\langle \zeta, \eta \rangle) \langle \eta, \zeta \rangle^k d\omega_2(\zeta) \right) \langle \xi, \eta \rangle^k, \qquad \xi \in \Omega_2.$$

The function $\xi \in \Omega_2 \to f(\xi)\bar{\xi}^k$ is positive definite on Ω_2 because it is a product of positive definite functions there. By Corollary 2.5, the integral

$$\int_{\Omega_2} f(\langle \zeta, \eta \rangle) \langle \eta, \zeta \rangle^k d\omega_2(\zeta)$$

is nonnegative and independent of η. Thus, we can write

$$f(\langle \xi, \eta \rangle) \sim \sum_{k \in \mathbb{Z}} a_k \langle \xi, \eta \rangle^k, \qquad a_k \geq 0,\ \xi, \eta \in \Omega_2.$$

In particular, we can write

$$f(\xi) \sim \sum_{k \in \mathbb{Z}} a_k \xi^k, \qquad a_k \geq 0,\ \xi \in \Omega_2.$$

To prove f coincides with the series it suffices to show that the series is absolutely convergent on Ω_2. But, letting $r \to 1^-$ in the inequality

$$\sum_{k=-m}^{m} a_k r^{|k|} |\xi| \leq \sum_{k=-m}^{m} a_k r^{|k|} \leq \sum_{k \in \mathbb{Z}} a_k r^{|k|}, \qquad m \in \mathbb{N},$$

and using Lemma 2.1, we obtain

$$\sum_{k=-m}^{m} a_k |\xi| \leq \sum_{k=-m}^{m} a_k \leq \lim_{r \to 1^-} \sum_{k \in \mathbb{Z}} a_k r^{|k|} = f(1), \qquad m \in \mathbb{N}.$$

Conversely, let f have a series representation as in the statement of the theorem. Taking an arbitrary positive integer l, complex numbers $c_1, c_2, \ldots, c_l$ and $\xi_1, \xi_2, \ldots, \xi_l$ in Ω_2 we have that

$$\begin{aligned} \sum_{\mu=1}^{l} \sum_{\nu=1}^{l} c_\mu \overline{c_\nu} f(\langle \xi_\mu, \xi_\nu \rangle) &= \sum_{k \in \mathbb{Z}} a_k \sum_{\mu=1}^{l} \sum_{\nu=1}^{l} c_\nu \overline{c_\nu} \langle \xi_\mu, \xi_\nu \rangle^k \\ &= \sum_{k \in \mathbb{Z}} a_k \left\langle \sum_{\mu=1}^{l} c_\mu \xi_\mu, \sum_{\nu=1}^{l} c_\nu \xi_\nu \right\rangle^k \\ &= \sum_{k \in \mathbb{Z}} a_k \left| \sum_{\mu=1}^{l} c_\mu \xi_\mu \right|^{2k} \geq 0. \end{aligned}$$

Therefore, f is positive definite on Ω_2. ∎

4. POSITIVE DEFINITE KERNELS ON Ω_{2q}, $q \geq 2$

In this section, we generalize Theorem 3.1 by characterizing the positive definite functions on Ω_{2q}, $q \geq 2$. For a continuous function f: $\Omega_{2q} \to \mathbb{C}$ we consider once again the solution of the Dirichlet problem on B_{2q}, i.e., we define

$$f(r\xi) := \int_{\Omega_{2q}} \mathscr{P}_q(r\xi, \eta) f(\eta)\, d\omega_{2q}(\eta), \qquad 0 \leq r < 1,\ \xi \in \Omega_{2q}.$$

Multiplying the equation in Lemma 2.3 by $f(\eta)$ and integrating we reach

$$\int_{\Omega_{2q}} \mathscr{P}_q(r\xi, \eta) f(\eta)\, d\omega_{2q}(\eta) = \sum_{(m,n)\in\mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} S^q_{m,n}(r) \times \int_{\Omega_{2q}} f(\eta) R^{q-2}_{m,n}(\langle \xi, \eta\rangle)\, d\omega_{2q}(\eta),$$

i.e.,

$$f(r\xi) = \sum_{(m,n)\in\mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} S^q_{m,n}(r) \int_{\Omega_{2q}} f(\eta) R^{q-2}_{m,n}(\langle \xi, \eta\rangle)\, d\omega_{2q}(\eta), \qquad 0 \le r < 1,\ \xi \in \Omega_{2q}.$$

Thus, on Ω_{2q}, we can write

$$f(\xi) \sim \sum_{(m,n)\in\mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} \int_{\Omega_{2q}} f(\eta) R^{q-2}_{m,n}(\langle \xi, \eta\rangle)\, d\omega_{2q}(\eta).$$

Next, we analyze the above identification for the case in which f is ζ-zonal. We define

$$\Psi(\xi) := \int_{\Omega_{2q}} f(\langle \eta, \zeta\rangle) R^{q-2}_{m,n}(\langle \xi, \eta\rangle)\, d\omega_{2q}(\eta), \qquad \xi \in \Omega_{2q},$$

and observe that Ψ is also ζ-zonal. Indeed, to see that, it suffices [9] to verify that $\Psi \circ B = \Psi$ for every unitary transformation B of $\mathbb{C}^q$ fixing ζ. But, for any such B,

$$\begin{aligned}\Psi(B\xi) &= \int_{\Omega_{2q}} f(\langle \eta, B\zeta\rangle) R^{q-2}_{m,n}(\langle B\xi, \eta\rangle)\, d\omega_{2q}(\eta)\\ &= \int_{\Omega_{2q}} f(\langle B^*\eta, \zeta\rangle) R^{q-2}_{m,n}(\langle \xi, B^*\eta\rangle)\, d\omega_{2q}(B^*\eta)\\ &= \int_{\Omega_{2q}} f(\langle \eta, \zeta\rangle) R^{q-2}_{m,n}(\langle \xi, \eta\rangle)\, d\omega_{2q}(\eta) = \psi(\xi), \qquad \xi \in \Omega_{2q}.\end{aligned}$$

Since the function $\xi \mapsto R^{q-2}_{m,n}(\langle \xi, \zeta\rangle)$ is an element of $\mathscr{H}^q_{m,n}$ so is Ψ. Due to Lemma 2.2, we conclude that

$$\Psi(\xi) = M P^{(q-2,|m-n|)}_{m\wedge n}(1) R^{q-2}_{m,n}(\langle \xi, \zeta\rangle), \qquad \xi \in \Omega_{2q},$$

for some constant M. The fact that $R^{q-2}_{m,n}(1) = 1$ allows us to infer that

$$\Psi(\zeta) = M P^{(q-2,|m-n|)}_{m \wedge n}(1).$$

Thus, we have proved the following result.

LEMMA 4.1. *For a continuous function f on B_2 and ξ, ζ on Ω_{2q}, the following expansion holds*:

$$f(\langle \xi, \zeta \rangle) \sim \sum_{(m,n)\in \mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} \left(\int_{\Omega_{2q}} f(\langle \eta, \zeta \rangle) R^{q-2}_{m,n}(\langle \zeta, \eta \rangle)\, d\omega_{2q}(\eta) \right) \times R^{q-2}_{m,n}(\langle \xi, \zeta \rangle).$$

We are now ready to prove the main result of this section.

THEOREM 4.2. *A continuous function f is positive definite on Ω_{2q} if and only if it is of the form*

$$f(z) = \sum_{(m,n)\in \mathbb{N}^2} a_{m,n} R^{q-2}_{m,n}(z), \qquad z \in B_2,$$

in which $\sum_{(m,n)\in \mathbb{N}^2} a_{m,n} < \infty$ and $a_{m,n} \geq 0$ for all (m,n).

Proof. First, assume that f is positive definite on Ω_{2q}. By Lemma 4.1, we may write

$$f(\langle \xi, \zeta \rangle) \sim \sum_{(m,n)\in \mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} a^{q-2}_{m,n}(\zeta) R^{q-2}_{m,n}(\langle \xi, \zeta \rangle), \quad \xi, \zeta \in \Omega_{2q},$$

in which

$$a^{q-2}_{m,n}(\zeta) := \int_{\Omega_{2q}} f(\langle \eta, \zeta \rangle) R^{q-2}_{m,n}(\langle \zeta, \eta \rangle)\, d\omega_{2q}(\eta).$$

By Corollary 2.5, $a^{q-2}_{m,n}$ is a nonnegative constant and, therefore, we can write

$$f(\langle \xi, \zeta \rangle) \sim \sum_{(m,n)\in \mathbb{Z}^2} a_{m,n} R^{q-2}_{m,n}(\langle \xi, \zeta \rangle), \qquad a_{m,n} \geq 0,\ \xi, \zeta \in \Omega_{2q}.$$

To see that $f(\langle \xi, \zeta \rangle)$ coincides with the series, it suffices to see that

$$\begin{aligned}\sum_{m,n=0}^{k} & a_{m,n} S^q_{m,n}(r) |R^{q-2}_{m,n}(\langle \xi, \zeta \rangle)| \\ & \leq \sum_{m,n=0}^{k} a_{m,n} S^q_{m,n}(r) |R^{q-2}_{m,n}(1)| \\ & \leq \sum_{(m,n)\in\mathbb{N}^2} a_{m,n} S^q_{m,n}(r) |R^{q-2}_{m,n}(1)|, \qquad 0 \leq r < 1,\ k \in \mathbb{N}.\end{aligned}$$

Letting $r \to 1^-$, we reach

$$\begin{aligned}\sum_{m,n=0}^{k} a_{m,n} |R^{q-2}_{m,n}(\langle \xi, \zeta \rangle)| & \leq \sum_{m,n=0}^{k} a_{m,n} |R^{q-2}_{m,n}(1)| \\ & \leq \lim_{r\to 1^-} \sum_{(m,n)\in\mathbb{N}^2} a_{m,n} S^q_{m,n}(r) |R^{q-2}_{m,n}(1)| \\ & = f(1),\end{aligned}$$

where the last inequality follows from Lemma 2.1. Thus, the series representing $f(\langle \xi, \zeta \rangle)$ is absolutely convergent in ξ and ζ showing that the function must coincide with the series. The desired representation for f is now visible.

Conversely, if f has the series representation as in the statement of the theorem, we may use the addition formula to reach

$$\begin{aligned}\sum_{\mu=1}^{l} \sum_{\nu=1}^{l} & c_\mu \overline{c_\nu} f\big(\langle \xi_\mu, \xi_\nu \rangle\big) \\ & = \sum_{(m,n)\in\mathbb{N}^2} a_{m,n} \frac{\omega_{2q}}{N(q;m,n)} \sum_{j=1}^{N(q;m,n)} \left| \sum_{\mu=1}^{l} c_\mu Y^q_j(\xi_\mu) \right|^2 \geq 0,\end{aligned}$$

whenever $l \geq 1$, $\{\xi_1, \xi_2, \ldots, \xi_l\} \subset \Omega_{2q}$ and $\{c_1, c_2, \ldots, c_l\} \subset \mathbb{C}$. Thus, f is positive definite on Ω_{2q}. ∎

5. ADDITIONAL CONSIDERATIONS

In this final section, we discuss one aspect on the connection between positive definiteness on Ω_{2q} and positive definiteness on real spheres. Let us denote by S^q the unit sphere in $\mathbb{R}^{q+1}$. Since Ω_{2q} contains a copy of S^{q-1} it follows at once that every positive definite function on Ω_{2q} is positive definite on S^{q-1}. What does not seem to be so obvious is that a positive definite function on Ω_{2q}, $q \geq 2$, can be positive definite on S^m, $q \leq m \leq 2q-1$. In order to explain that this can happen, we need the following result linking ultraspherical polynomials and disk polynomials discovered by Dreseler and Hrach [4].

LEMMA 5.1. *For any $\alpha \geq 0$ and any nonnegative integer k, there are positive constants $\{d(k, \alpha, m, n)\colon\ m + n = k\}$ such that $d(k, \alpha, m, n) = d(k, \alpha, n, m)$ and*

$$R_k^{(\alpha+1/2,\,\alpha+1/2)}(\operatorname{Re} z) = \sum_{m+n=k} d(k, \alpha, m, n) R_{m,n}^{\alpha}(z), \qquad z \in B_2.$$

We observe that the coefficients $d(k, \alpha, m, n)$ can be computed by the formula

$$d(k, \alpha, m, n) = \left(\int_{-1}^{1} \left| R_k^{(\alpha+1/2,\,\alpha+1/2)}(x) \right|^2 dw_{\alpha+1/2,\,\alpha+1/2}(x) \right) \times \left(\int_{B_2} \left| R_{m,n}^{\alpha}(z) \right|^2 dw_{\alpha}(z) \right)^{-1},$$

where $dw_{\alpha+1/2,\,\alpha+1/2}(x) = c(\alpha)(1-x^2)^{\alpha+1/2}dx$ and $c(\alpha)$ is a constant chosen so that the measure has total mass equal to one.

Let $g\colon [-1,1] \to \mathbb{R}$ be a positive definite function on S^{2q-1}, $q \geq 2$. According to Schoenberg's theorem,

$$g(x) = \sum_{k \in K} a_k R_k^{(q-3/2,\,q-3/2)}(x), \qquad K \subset \mathbb{N},\ a_k \geq 0,\ \sum_{k \in K} a_k < \infty.$$

By Lemma 5.1, we obtain

$$\begin{aligned} g(\operatorname{Re} z) &= \sum_{k \in K} a_k \sum_{m+n=k} d(k, q-2, m, n) R_{m,n}^{q-2}(z) \\ &= \sum_{k \in K} \sum_{m+n=k} a_k d(k, q-2, m, n) R_{m,n}^{q-2}(z) \\ &= \sum_{m+n \in K} a_{m+n} d(m+n, q-2, m, n) R_{m,n}^{q-2}(z), \qquad z \in B_2. \end{aligned}$$

Due to the assumptions on g, the resulting sum defines a real positive definite function f on Ω_{2q}. Since g is the restriction of f to $[-1,1]$, it follows that f is positive definite on S^{2q-1}. Thus, we have concluded that there are positive definite functions on Ω_{2q} that are positive definite not only on S^{q-1} but also on all real spheres of dimension up to $2q$.

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